Problem One.

Let $S \subset \mathbb{R}^n$ be a measurable set with $m(S) < \infty$ and let $\epsilon > 0$ be a positive real number. Show that there exists a compact subset $K \subset S$ such that

$$m(S) - \epsilon \leq m(K) \leq m(S).$$

Proof. If S is measurable, then $m(S) = m^*(S) = \inf_{O \supset S} m^*(O)$, so for any $\epsilon > 0$ there exists some open $O_\epsilon \supset S$ for which $m(O_\epsilon) - m(S) < \epsilon$.

Now in fact we want to apply this Remark not to S but to S^c. Notice that an open set $O_\epsilon \supset S^c$ has as its complement a closed set $(O_\epsilon)^c = F_\epsilon \subset S$.

Therefore, for any $\epsilon > 0$ we can find $F_{\epsilon/2} \subset S$ that $m(S) - m(F_{\epsilon/2}) < \frac{\epsilon}{2}$.

Now to generate a compact set K, note that a compact set in $\mathbb{R}^n$ is the same as a closed and bounded set. Therefore, for fixed ϵ, let $K_n := \overline{B_n(0)} \cap F_{\epsilon/2}$, where $\overline{B_n(0)}$ denotes the closed ball of radius n and centre 0. Now thus defined, K_n is compact for all $n \in \mathbb{N}$,

$$K_1 \subset K_2 \subset K_3 \subset \cdots \subset F_{\epsilon/2},$$

and

$$F_{\epsilon/2} = \bigcup_{n=1}^{\infty} K_n.$$

By monotonicity of measure and by the monotone convergence theorem in $\mathbb{R}$, it follows that

$$m(F_{\epsilon/2}) = \lim_{n \to \infty} m(K_n)$$

and for any $\epsilon > 0$ there exists N sufficiently large that

$$m(F_{\epsilon/2}) - \frac{\epsilon}{2} < m(K_N) < m(F_{\epsilon/2}).$$

Therefore,

$$m(S) - \epsilon < m(K_N) < m(S)$$

as desired. □

Problem Two.

Let X be a set and $\mathcal{M}$ be a $\sigma-$algebra of subsets of X. Suppose that $m : \mathcal{M} \to [0, \infty]$ is a finitely additive, countably subadditive function. Show that m is countably additive.

Proof. It suffices to show that for any pairwise disjoint sequence of subsets $S_0, S_1, S_2, \cdots \subset X$, and for any $N \in \mathbb{N}$,

$$\sum_{k=1}^{N} m(S_k) \leq m(S)$$

where $S = \bigcup_{k=1}^{\infty} S_k$. But indeed, write S as a disjoint union

$$S = \left(\bigcup_{k=1}^{N} S_k \right) \cup \left(\bigcup_{k=N+1}^{\infty} S_k \right)$$

so that by finite additivity,

$$m(S) = m\left(\bigcup_{k=1}^{N} S_k \right) + m\left(\bigcup_{k=N+1}^{\infty} S_k \right) = \sum_{k=1}^{N} m(S_k) + m\left(\bigcup_{k=N+1}^{\infty} S_k \right) \geq \sum_{k=1}^{N} m(S_k)$$

as required. □

Appendix to Problem One.

I typed out this unnecessary lemma before realising it was unnecessary for this problem. Nevertheless, it is interesting and important.

Lemma. If S is measurable, then for all $\epsilon > 0$ there exists an open $O_\epsilon \supset S$ such that $m^*(O_\epsilon - S) < \epsilon$.

Indeed, if S is measurable, then for any $A \in \mathbb{R}^n$,

$$m^*(A) = m^*(A \cap S) + m^*(A \cap S^c).$$

We also know that $m^*(S) = \inf_{O \supset S} m^*(O)$, so for any $\epsilon > 0$ there exists some open $O_\epsilon \supset S$ for which $m^*(O_\epsilon) - m^*(S) < \epsilon$. Put $A = O_\epsilon$ into the equation above, to obtain

$$m^*(O_\epsilon) = m^*(O_\epsilon \cap S) + m^*((O_\epsilon)^c \cap S)$$

which is

$$m^*(O_\epsilon) = m^*(S) + m^*((O_\epsilon)^c - S).$$

Case 1. Suppose that $m^*(S) = m(S) < \infty$. Then we can subtract it from both sides to get

$$m^*((O_\epsilon)^c - S) = m^*(O_\epsilon) - m^*(S) < \epsilon$$

as required.

Case 2. Alternatively, if $m(S) = \infty$, then chop up S into countably many pieces $S_1, S_2, S_3, \cdots$ each of finite measure, and apply Case 1 to each piece, choosing $\epsilon_j = 2^{-j}\epsilon$ for S_j. End of Lemma. □

Remark. This Lemma furnishes us with a very important alternative definition of what it means to be a measurable set:

Definition. $S \subset \mathbb{R}^n$ is *measurable* just if for all $\epsilon > 0$ there exists an open $O_\epsilon \supset S$ such that $m^*(O_\epsilon - S) < \epsilon$.

N.B. that I haven't proved the back direction, namely that this definition entails the prof's definition of "measurable". The prof's definition is usually known as the *Caratheodory definition* of measurability. The back direction is "left as exercise." End of Remark.